THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 15
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1 Lagrange Duality for Convex Optimization

Recall that

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} g_i(x) \le 0, & i = 1,\dots,\ell \\ h_j(x) = 0, & j = 1,\dots,m \end{cases} \tag{P}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex differentiable functions while $h_j : \mathbb{R}^n \to \mathbb{R}$ are affine functions, i.e. $h_j(x) = A_j^T x + b_j$.

Definition 1. Lagrangian function L is defined by

$$L(x,\lambda,\mu) := f(x) + \sum_{i=1}^{\ell} \lambda_i g_i(x) + \sum_{j=1}^{m} \mu_j h_j(x)$$

for $\lambda_i \ge 0$, $\mu_j \in \mathbb{R}$ for all $i = 1,\dots,\ell$ and $j = 1,\dots,m$.

Remarks. We have

$$(P) \iff \min_{x\in\mathbb{R}^n} \sup_{\substack{\lambda_i\in\mathbb{R}_+ \\ \mu_j\in\mathbb{R}}} L(x,\lambda,\mu)$$

since $\sup_{\substack{\lambda_i\in\mathbb{R}_+ \\ \mu_j\in\mathbb{R}}} L(x,\lambda,\mu) = \begin{cases} f(x) & \text{if } g_i(x) \le 0,\ h_j(x) = 0 \\ +\infty & \text{otherwise} \end{cases}$

Definition 2. The Lagrange dual problem is defined as:

$$\max_{\substack{\lambda_i\in\mathbb{R}_+,\ i=1,\dots,\ell \\ \mu_j\in\mathbb{R},\ j=1,\dots,m}} d(\lambda,\mu)$$

where $d(\lambda,\mu) := \inf_{x\in\mathbb{R}^n} L(x,\lambda,\mu)$.

Weak Duality

Proposition 1. *Given any non-empty sets A, B and a function L defined on $\mathbb{R}$, then*

$$\inf_{a\in A} \sup_{b\in B} L(a,b) \ge \sup_{b\in B} \inf_{a\in A} L(a,b)$$

Proof. Since

$$L(a,b) \ge \inf_{\tilde{a}\in A} L(\tilde{a},b), \quad \forall a\in A, b\in B$$

This implies that

$$\sup_{b\in B} L(a,b) \ge \sup_{b\in B} \inf_{\tilde{a}\in A} L(\tilde{a},b), \quad \forall a \in A$$

We treat the right-hand-side as a function C and left-hand-side as $f(a)$, then

$$f(a) \geq C, \ \ \forall a \in A$$

Taking infimum over A yields $\inf_{a\in A} f(a) \geq C$, that is

$$\inf_{a\in A} \sup_{b\in B} L(a,b) \geq \sup_{b\in B} \inf_{\tilde{a}\in A} L(\tilde{a},b).$$

$\square$

Theorem 2. *Let f, g_i be convex and differentiable, and h_j be affine. Assume that $f^* > -\infty$ is bounded below, and there exist $\bar{x} \in \mathbb{R}^n$ such that $g_i(\bar{x}) < 0$ for all $i = 1, \ldots, \ell$ and $h_j(\bar{x}) = 0$ for all $j = 1, \ldots, m$, then the Lagrange dual is solvable and we have (λ^*, μ^*) such that*

$$d(\lambda^*, \mu^*) = \max_{\substack{\lambda_i \in \mathbb{R}_+,\ i=1,\ldots,\ell \\ \mu_j \in \mathbb{R},\ j=1,\ldots,m}} d(\lambda, \mu) = f^*.$$

1.1 Proof of Strong Duality

Consider the case without equality constraint first, that is $h_j(x) = 0$.
Let us conisder the system of constraints:

$$\begin{cases} f(x) < c \\ g_i(x) \leq 0, \quad i = 1, \ldots, \ell \end{cases} \tag{I}$$

$$\begin{cases} \lambda_i \geq 0. \ \ i = 1, \ldots, \ell \\ \inf\limits_{x} \left(f(x) + \sum\limits_{i=1}^{m} \lambda_i g_i(x) \right) \geq C \end{cases} \tag{II}$$

Proposition 3. *If there exists $\lambda_i, \ i = 1, \ldots, \ell$ satisfying (II), then there is no $x \in \mathbb{R}^n$ satisfying (I).*

Proof. Assume that (II) is solvable, then there exists $\lambda_i \geq 0$ such that

$$\inf_{x\in\mathbb{R}^n} \left(f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right) \geq C$$

i.e. $f(x) + \sum\limits_{i=1}^{m} \lambda_i g_i(x) \geq C$ for all $x \in \mathbb{R}^n$. Next, if $x \in \mathbb{R}^n$ satisfies $g_i(x) \leq 0$, then

$$f(x) \geq C - \sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{g_i(x)}_{\leq 0} \geq C$$

and thus (I) is insolvable. $\square$

Remarks. Even we consider adding the equality constraints, the proof is still similar.

Proposition 4. *If (I) is insolvable, and the subsystem*

$$\begin{cases} g_j(\bar{x}) < 0, \ i = 1, \ldots, \ell \\ h_j(\bar{x}) = 0, \ j = 1, \ldots, m \end{cases}$$

is solvable, then (II) is solvable.

Remarks. (I) is insolvable means that for all $x \in \mathbb{R}^n$ satisfying the constraints $g_i(x) \leq 0, \ h_j(x) = 0$ for all $i = 1, \ldots, \ell, \ j = 1, \ldots, m$, then implies that $f(x) \geq C$.

— End of Lecture 15 —